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On the finite size corrections of anti-ferromagnetic anomalous dimensions in $\mathcal{N} = 4$ SYM

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Abstract: Non-linear integral equations derived from Bethe Ansatz are used to evaluate finite size corrections to the highest (i.e. *anti-ferromagnetic*) and immediately lower anomalous dimensions of scalar operators in $\mathcal{N} = 4$ SYM. In specific, multi-loop corrections are computed in the SU(2) operator subspace, whereas in the general SO(6) case only one loop calculations have been finalised. In these cases, the leading finite size corrections are given by means of explicit formulæ and compared with the exact numerical evaluation. In addition, the method here proposed is quite general and especially suitable for numerical evaluations.

Keywords: Bethe Ansatz, Lattice Integrable Models, Duality in Gauge Field Theories.

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1. Prologue

The AdS/CFT correspondence relates string states and local gauge-invariant operators of a dual quantum field theory [1]. The energies of the string states correspond to the eigenvalues (the so-called anomalous dimensions) of the mixing matrix of gauge field theory operators. However, even for the well understood case of the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, testing of such a correspondence has revealed to be a rather difficult task. Indeed perturbative expansions in SYM assume the 't Hooft coupling λ to be small, while string perturbation theory makes sense when the string tension $T = \sqrt{\lambda}/2\pi$ is large. A conceptual progress came with the proposal [2] of restricting to states (named afterwards BMN states) represented by small closed strings with large angular momentum J. Indeed, the energy of these states admits an expansion in the small classical parameter λ/J^2, the quantum corrections corresponding to another $1/J$ expansion. On the SYM side, these string states correspond to composite operators containing L scalars. However, this implies that the (for now) accessible semiclassical regime of string states is mimicked in the gauge theory by "long" operators, which renders a priori the computation of their anomalous dimensions a complex problem. In this perspective, it can be understood why a very important progress was realised by Minahan and Zarembo while noticing the coincidence

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between the one-loop expansion of the mixing matrix for operators in the $\mathcal{N}=4$ SYM theory containing L scalars and an integrable SO(6), L site spin chain Hamiltonian [3]. For, since then, many Bethe Ansatz ideas and techniques were used in order to find anomalous dimensions of very long operators of gauge theories, as long as this novel *correspondence* between Integrable Models (IMs) and (S)YM theories was extended to the full $PSU(2,2|4)$ symmetry [4], to higher loop expansions [5−7] and to other gauge theories [8−12]. For completeness' sake, we should also mention how this IM/SYM relation makes more complete and deeper the previous appearance of an integrable system in the barely perturbative regime of QCD [13, 14]. In particular this link to IMs stimulated an impressive activity, which allowed many scholars to test the AdS/CFT duality in different cases (e.g. for the BMN operators of [2] and for others too). More precisely, the integrability of the mixing matrix at all orders in perturbation theory was conjectured in [5, 7] and then proved for the SU(2) subsector up to three loops in [6]. In this paper, the dilatation operator was embedded into a long-range spin chain, the Inozemtsev spin chain [15]. However, the Inozemtsev spin chain at four loops would lose the apparently desirable property (in perturbative gauge theory) of BMN scaling and this lack stimulated the conjecture according to which an alternative long-range spin chain for all number of loops may exist [7]. A Bethe Ansatz was also proposed in [7] for all the values of the coupling g (cf. below), in order to deal with this otherwise unknown multi-loop Hamiltonian. However, the intriguing recent paper [16] has pointed out many reasons why this Bethe Ansatz may not furnish the right anomalous dimension at and beyond the wrapping order g^{2L} (note also [17] for a qualitative interpretation). The same paper has identified in the half filled Hubbard model a short range Hamiltonian conjectured to reproduce the mixing matrix. Actually, this identification was explicitly carried out up to three loops and it is unclear if it may survive the break-down order g^{2L}.

Despite the great amount of work on the subject, the majority of the results found up to now (with exception of very few papers like, for instance, [18−20]) concerns the calculation of only the leading term of the anomalous dimensions of arrays of L operators in the limit $L \to \infty$. In the BMN sector, this corresponds to classical energies on the string side. Consequently, the correction to this leading term is related to quantum fluctuations of the string state energy and is therefore worth studying. In this respect and from the Bethe Ansatz point of view the $L \to \infty$ limit may be described in all its physical quantities in terms of the density of Bethe roots (per quantum numbers), provided the latter really tends to a continuous distribution when $L = \infty$ (cf. [21] for some remarks on this point). In any event, almost as early as the Bethe's invention [22] (for the spectrum of the isotropic Heisenberg spin 1/2 chain), a linear integral equation constraining the density (for the antiferromagnetic ground state) was derived and solved [23]. Since those early stages the power and versatility of Bethe Ansatz was being very much appreciated, at most in condensed matter physics, integrable models theory[1] and statistical mechanics (cf. [25, 26] just for some examples). Also, the integral equation idea lived a revival since 1964 [27, 24] and was

[1]The first liaison with the quantum version of integrability, the Yang-Baxter equation, was found by the seminal contributions [24].

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extended to excited states (cf. the profundus review [28]) and to the statistical view [29]. Implementing this latter in the framework of the relativistic factorised scattering theory, Al. Zamolodchikov formulated a general and pretty widely applicable idea concerning an exact formula of the vacuum scaling function at all size scales, the so-called thermodynamic Bethe Ansatz [30].

As for finite size effects in quantum integrable systems, the Non-Linear Integral Equation (NLIE) description — first introduced in [31] for the conformal (anti-ferromagnetic) vacuum and then derived for an off-critical vacuum in [32] by other means — turned out to be an efficient tool in order to explore the scaling properties of the energy. Since [33], regarding excitations on the vacuum, a number of articles was devoted to the analysis of and through a NLIE and mainly follows the route pioneered by Destri and de Vega [32] (cf. the lectures [34] for an overview). In this way (which will be ours too), the NLIE stems directly from the Bethe equations and characterises a quantum state by means of a single (or very few) integral equation(s) in the complex plane (and possibly some auxiliary algebraic equations). The NLIE has been widely studied for integrable models described by trigonometric-type Bethe equations: for instance, the 1/2-XXZ spin chain [31], the inhomogeneous 1/2-XXZ and sine-Gordon field theory (ground state in [32], excited states in [33]) and the quantum (m)KdV/sine-Gordon theory [35].

In this paper, we want to propose the Non-Linear Integral Equation idea [31 – 34, 36] as a tool to compute finite L corrections to the anomalous dimensions of (long) operators in $\mathcal{N} = 4$ SYM. In terms of the solution of the NLIE, we can indeed write down exact expressions for the *observable* eigenvalues, as they depend on the Bethe roots. Their behaviour for large L may be disentangled analytically and numerically. Going into more details, we will concentrate our analysis here on the operator with the highest anomalous dimension in the physical sector of array operators made up of a fixed number L of scalars (without derivatives neither fermions[2]). This corresponds in the spin chain to the *anti-ferromagnetic* state, made up of a sea of real Bethe roots. We will also study *excitations*[3] thereof, introduced by the presence of holes. These are the simplest possible modifications, as already argued in [33], though the anti-ferromagnetic state is not the (true) vacuum (with smallest energy or anomalous dimension) of the chain, which enjoys a ferromagnetic nature and corresponds, in the gauge theory parlance, to the BPS state with all the partons (i.e. the complex scalars) of the same kind. On the contrary, it becomes of interest here as its eigenvalue constitutes the upper bound, i.e. the largest anomalous dimension: the finiteness of the spectrum is very clear in the spin chain and SYM interpretation, although a momentum bound of the string is rather not obvious (but semiclassical computation can be trusted in this regime just partially and have been started recently [37]). Moreover, it plays the rôle of the genuine vacuum in the large N QCD expansion, at least at one loop (cf., for instance, [11]). Its interest resides also in the fact that the holes excitations will furnish the just smaller anomalous dimensions, whose energies are neglected in condensed matter

[2]Closure of this sector under renormalisation is proved up to one loop for the entire scalar sector (SO(6) case) and for all loops for two scalars (SU(2) case); cf. below.

[3]This terminology is borrowed from the cases when the anti-ferromagnetic configuration yields the (true) vacuum.

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physics since this part of the spectrum decouples to infinity from the *real* spectrum above the ferromagnetic vacuum, in the thermodynamic limit. In this respect, we also emphasise that the Néel state ($|\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow\rangle$) is not an eigenstate in this context. Specifically, we will study both the general SO(6) case (at one loop) and the SU(2) subcase (though this with an arbitrary number of loops), providing exact expressions for the anomalous dimensions of SYM operators with finite number (L) of fields. In fact, as a starting point for the latter we shall use the asymptotic Bethe Ansatz of [7], whose reliability is up to order g^{2L-2}, as already widely stressed. For clarity and simplicity reasons, we will start by the exposition of the SU(2) case which, after the proposal by [16], may be interpreted as a strong coupling expansion of the Hubbard model at *all* orders (provided L is large enough). We will introduce the techniques in the known example of the Heisenberg chain and we will provide original results for the many loop Bethe Ansatz of [7], as well as for the SO(6) case.

2. The SU(2) case: one loop or the Heisenberg chain

Let us first consider the SU(2) subsector of the gauge-invariant scalar operators in $\mathcal{N}=4$ SYM field theory. The anomalous dimension of a general composite operator containing L scalars is given by

$$\gamma = \frac{\lambda}{8\pi^2} E \,, \tag{2.1}$$

where $\lambda = N g^2_{\mathrm{YM}} = 8\pi^2 g^2$ is the 't Hooft coupling of the SU(N) super Yang-Mills theory and

$$E = \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}} \tag{2.2}$$

is the energy of a spin 1/2-XXX chain, i.e. the celebrated Heisenberg spin chain, with L sites. Since the pioneering work of Bethe [22] it is well known that the M complex numbers (or Bethe roots) u_k must satisfy the equations

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{u_j - u_k - i}{u_j - u_k + i} \,, \quad j = 1, \ldots, M \,, \tag{2.3}$$

usually named after Bethe as well. In this approach, one set $\{u_k\}$ of solutions identifies one energy eigenfunction. In the original paper the previous equations are the consequence of the imposition of periodicity of the postulated wave function (the famous Ansatz), without any clear mention to integrability. This Bethe eigenfunction is also (highest weight) eigenstate of the total z-component spin operator with integer or half-integer eigenvalue $S = L/2 - M \geqslant 0$.

Now, we derive a single Non-Linear Integral Equation (NLIE) along the ideas of [32, 33], so that we may have in it a more effective, though equivalent, description of Bethe equations. We may need to say that this derivation will have a pedagogical purpose in perspective of the multi-loop case of next section, though it will help to illustrate the general idea of [32, 33] and to interpret the results from the gauge theory viewpoint. In fact, on the one hand it is just a limiting case of the *general* Bethe Ansatz of the next section, on the other hand similar results are already contained in [31, 36].

The NLIE will be an equation for the so-called counting function,

$$Z(x) = L\phi\left(x, \frac{1}{2}\right) - \sum_{k=1}^{M} \phi(x - u_k, 1)\,, \tag{2.4}$$

where the function

$$\phi(x, \xi) = i \ln\left(\frac{i\xi + x}{i\xi - x}\right) = 2 \arctan \frac{x}{\xi}\,, \quad \xi > 0\,, \tag{2.5}$$

is analytic in the strip $|\mathrm{Im} x| < \xi$ provided the branch of the logarithm is along the negative real axis. Then we need a variable that keeps into account the parity of the chain in relation with the number of Bethe roots:

$$\delta = (L - M) \bmod 2\,. \tag{2.6}$$

After, by using the simple property

$$i \ln \frac{x - i\xi}{x + i\xi} - i \ln \frac{i\xi - x}{i\xi + x} = \pi\,, \tag{2.7}$$

the Bethe equations can be written in the form

$$iL \ln \frac{\frac{i}{2} + u_j}{\frac{i}{2} - u_j} - \sum_{k=1}^{M} i \ln \frac{i + u_j - u_k}{i - u_j + u_k} = \pi(2I_j + \delta - 1)\,, \quad j = 1, \ldots, M\,,$$

thanks to the introduction of certain integer quantum numbers I_j, or in terms of the counting function as

$$Z(u_j) = \pi(2I_j + \delta - 1)\,, \quad j = 1, \ldots, M\,. \tag{2.8}$$

The last equations are completely equivalent to the initial Bethe ones (2.3), provided that u_j enter the counting function by (2.4).

From now on and only for simplicity reasons we will be considering states characterised by real roots. This is the formulation proposed in [33] and it can be easily extended to deal with arbitrary complex roots. Bearing in mind the limits

$$\lim_{x \to \pm\infty} \phi(x, \xi) = \pm\pi\,, \tag{2.9}$$

we easily compute the limiting values of the counting function

$$\lim_{x \to \pm\infty} Z(x) = \pm(L - M)\pi\,.$$

Since $Z(x)$ is an increasing function, the condition (2.8) is satisfied by $L - M$ points on the real axis, among which there are indeed M Bethe roots. The number of the remaining *fake*[4] solutions (holes) is

$$H = L - M - M = L - 2M\,. \tag{2.10}$$

[4]This in the obvious sense that these are **not** solutions of the initial Bethe equations (2.3).

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Of course, the holes x_h are determined by the same equations as those for real roots, but with the complementary set of integer quantum numbers I_h, namely

$$Z(x_h) = \pi(2I_h + \delta - 1)\,, \tag{2.11}$$

since holes do not satisfy the Bethe equations. Hence, both Bethe roots and holes, respectively, enjoy the condition

$$\exp[iZ(x)] = (-1)^{\delta-1}\,, \quad x = u_j, x_h\,. \tag{2.12}$$

Now, let $f(u)$ be an analytic function within a strip around the real axis. Thanks to (2.12), the sum of its values at all the Bethe roots takes on the expression [33]

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) \,=\, &-\int_{-\infty}^{\infty}\frac{dx}{2\pi i}f'(x-i\epsilon)\,\ln\left[1+(-1)^{\delta}e^{iZ(x-i\epsilon)}\right] - \\ &-\int_{\infty}^{-\infty}\frac{dx}{2\pi i}f'(x+i\epsilon)\,\ln\left[1+(-1)^{\delta}e^{iZ(x+i\epsilon)}\right] - \sum_{h=1}^{H} f(x_h)\,,\end{aligned} \tag{2.13}$$

with $\epsilon > 0$ small enough to keep the integration inside the analyticity strip. If now $\epsilon \to 0$, we may rearrange this expression as

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) \,=\, &-\int_{-\infty}^{\infty}\frac{dx}{2\pi}f'(x)Z(x) + \\ &+\int_{-\infty}^{\infty}\frac{dx}{\pi}f'(x)\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(x+i0)}\right] - \sum_{h=1}^{H} f(x_h)\,.\end{aligned} \tag{2.14}$$

Upon applying (2.14) to the sum over the Bethe roots in the definition of the counting function (2.4), we obtain yet a first integral equation for it,

$$\begin{aligned}Z(x) \,=\, &L\phi\left(x,\frac{1}{2}\right) - \int_{-\infty}^{\infty}\frac{dy}{2\pi}\phi'(x-y,1)Z(y) + \\ &+\int_{-\infty}^{\infty}\frac{dy}{\pi}\phi'(x-y,1)\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0)}\right] + \\ &+\sum_{h=1}^{H}\phi(x-x_h,1)\,.\end{aligned} \tag{2.15}$$

As usual, we introduce a shorthand

$$L(x) = \mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(x+i0)}\right]\,, \tag{2.16}$$

and then take the Fourier transform[5] of all terms in (2.15) to obtain

$$\hat{Z}(k) = L\hat{\phi}\left(k,\frac{1}{2}\right) - \frac{1}{2\pi}\hat{\phi}'(k,1)\hat{Z}(k) + \frac{1}{\pi}\hat{\phi}'(k,1)\hat{L}(k) + \sum_{h=1}^{H} e^{-ikx_h}\hat{\phi}(k,1)\,. \tag{2.18}$$

[5]We define the Fourier transform $\hat{f}(k)$ of a function $f(x)$ as given by

$$\hat{f}(k) = \int_{-\infty}^{\infty} dx\; e^{-ikx} f(x)\,. \tag{2.17}$$

This equation can be recast in the more compact form

$$\hat{Z}(k) = \hat{F}(k) + 2\hat{G}(k)\hat{L}(k) + \sum_{h=1}^{H} e^{-ikx_h}\hat{H}(k)\,,$$

where the Fourier transform of the forcing term reads as

$$\hat{F}(k) = L\frac{\hat{\phi}\left(k,\frac{1}{2}\right)}{1+\frac{1}{2\pi}\hat{\phi}'(k,1)}\,, \tag{2.19}$$

that of the kernel as

$$\hat{G}(k) = \frac{\frac{1}{2\pi}\hat{\phi}'(k,1)}{1+\frac{1}{2\pi}\hat{\phi}'(k,1)}\,, \tag{2.20}$$

and eventually the holes contribution is (P is the principal value distribution)

$$\hat{H}(k) = \frac{\hat{\phi}(k,1)}{1+\frac{1}{2\pi}\hat{\phi}'(k,1)} = \frac{2\pi}{i}P\left(\frac{1}{k}\right)\hat{G}(k)\,. \tag{2.21}$$

All these can be easily calculated, once the Fourier transform of the function

$$\phi'(x,\xi) = \frac{2\xi}{\xi^2+x^2}\,, \tag{2.22}$$

is explicitly computed as

$$\hat{\phi}'(k,\xi) = 2\pi e^{-\xi|k|}\,, \tag{2.23}$$

which entails

$$\hat{F}(k) = LP\left(\frac{1}{k}\right)\frac{2\pi e^{-\frac{|k|}{2}}}{i(1+e^{-|k|})}\,, \quad \hat{G}(k) = \frac{1}{1+e^{|k|}}\,. \tag{2.24}$$

Upon anti-transforming, we obtain the forcing term

$$F(x) = L\int_0^\infty \frac{dk}{k}\,\frac{\sin kx}{\cosh\frac{k}{2}} = 2L\arctan e^{\pi x} - \frac{L\pi}{2} = L\,\mathrm{gd}\,\pi x\,,$$

and besides the kernel

$$G(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,e^{ikx}\frac{1}{1+e^{|k|}} = \frac{1}{2\pi i}\frac{d}{dx}\ln\frac{\Gamma\left(1+\frac{ix}{2}\right)\Gamma\left(\frac{1}{2}-\frac{ix}{2}\right)}{\Gamma\left(1-\frac{ix}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{2}\right)} = \frac{1}{2\pi i}\frac{d}{dx}\ln S(x)\,,$$

where the expression in terms of Euler's gamma functions,

$$S(x) = \frac{\Gamma\left(1+\frac{ix}{2}\right)\Gamma\left(\frac{1}{2}-\frac{ix}{2}\right)}{\Gamma\left(1-\frac{ix}{2}\right)\Gamma\left(\frac{1}{2}+\frac{ix}{2}\right)}\,,$$

is indeed *the scattering factor* of the NLIE (cf. [33, 38, 39] for a justification of this name). Finally, from this we can easily gain the hole function in the form

$$H(x) = 2\pi\int_0^x dy\,G(y) = -i\ln S(x)\,.$$

With all these functions at hand, we are in the position to write down the announced non-linear integral equation for $Z(x)$,

$$Z(x) = F(x) - i \sum_{h=1}^{H} \ln S(x - x_h) + 2 \int_{-\infty}^{\infty} dy\, G(x - y) \,\mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] , \quad (2.25)$$

and we can also check *a posteriori* that no zero-modes actually entered its derivation.

The NLIE (2.25) together with the holes quantization conditions (2.11) is equivalent to the Bethe equations (2.3).

2.1 The *observable* eigenvalues.

Let us now pass on to the computation of the eigenvalues of the *observables* on states containing M real Bethe roots and H holes. We move from

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) = &- \int_{-\infty}^{\infty} \frac{dx}{2\pi} f'(x) Z(x) + \\ &+ \int_{-\infty}^{\infty} \frac{dx}{\pi} f'(x) \,\mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(x+i0)}\right] - \sum_{h=1}^{H} f(x_h)\end{aligned} \quad (2.26)$$

and then insert into this expression the non-linear integral equation (2.25) and re-organise the terms as

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) = &- \int_{-\infty}^{\infty} \frac{dx}{2\pi} f'(x) F(x) + \sum_{h=1}^{H} \left\{ \int_{-\infty}^{\infty} \frac{dx}{2\pi} f'(x)\; i \ln S(x - x_h) - f(x_h) \right\} + \\ &+ \int_{-\infty}^{\infty} \frac{dx}{\pi} f'(x) \int_{-\infty}^{\infty} dy\; [\delta(x-y) - G(x-y)] \,\mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] .\end{aligned} \quad (2.27)$$

This formula gives an exact expression for the eigenvalues of any general observable in terms of the solution of the Non-Linear Integral Equation, solution which characterises the specific eigenstate. For example, its analogue was exploited in the quantum (m)KdV context [35] to obtain the quintessence of an integrable model, namely the (commuting) integrals of motion. Now, we want to use it in order to compute the eigenvalues of the energy (anomalous dimension) and of the momentum.

2.2 The anomalous dimension

As illustrated in (2.2), to compute the (total) energy, we need to apply formula (2.27) with the single particle energy

$$f(x) \equiv e(x) \equiv \frac{1}{x^2 + \frac{1}{4}} . \quad (2.28)$$

Indeed, the first term of the l.h.s. is given by

$$\begin{aligned}- \int_{-\infty}^{\infty} \frac{dx}{2\pi} e'(x) F(x) &= \int_{-\infty}^{\infty} \frac{dx}{2\pi} e(x) F'(x) = L \int_{-\infty}^{\infty} \frac{dx}{2} \frac{1}{x^2 + \frac{1}{4}} \frac{1}{\cosh \pi x} \\ &= L \int_{-\infty}^{\infty} dy \frac{1}{y^2 + 1} \frac{1}{\cosh \frac{\pi y}{2}} = 2L \ln 2 .\end{aligned} \quad (2.29)$$

The last term reads

$$\int_{-\infty}^{\infty} \frac{dx}{\pi} e'(x) \int_{-\infty}^{\infty} dy \, [\delta(x-y) - G(x-y)] \, \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] ,$$

where the x-convolution is conveniently evaluated in Fourier space (where it becomes an ordinary product):

$$\int_{-\infty}^{\infty} \frac{dy}{\pi} \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] \frac{d}{dy} \int_{-\infty}^{\infty} dk \frac{e^{iky}}{2\cosh\frac{k}{2}} =$$
$$= \int_{-\infty}^{\infty} dy \left(\frac{d}{dy} \frac{1}{\cosh \pi y}\right) \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] . \tag{2.30}$$

Eventually, we need to compute the two terms of the hole sum (the second term):

$$\sum_{h=1}^{H} \left\{ \int_{-\infty}^{\infty} \frac{dx}{2\pi} e'(x) \; i \ln S(x - x_h) - e(x_h) \right\} .$$

For the first of them we may write

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} \, e'(x) \, i \ln S(x - x_h) = \int_{-\infty}^{\infty} dx \, e(x) \, G(x - x_h) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx_h} \, \hat{e}(k) \, \hat{G}(k) \, .$$

This yields, once the second term is expressed by its Fourier transform,

$$\sum_{h=1}^{H} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx_h} \, \hat{e}(k) \, [\hat{G}(k) - 1] = -\sum_{h=1}^{H} \int_{-\infty}^{\infty} dk \, \frac{e^{ikx_h}}{2\cosh\frac{k}{2}} \, , \tag{2.31}$$

where we have used the formula

$$\hat{e}(k) = 2\pi e^{-\frac{1}{2}|k|} \, ,$$

particular case of (2.23), and the expression of $\hat{G}(k)$, (2.24). Eventually, the source term may be written as

$$-\sum_{h=1}^{H} \int_{-\infty}^{\infty} dk \, \frac{e^{ikx_h}}{2\cosh\frac{k}{2}} = -\sum_{h=1}^{H} \frac{\pi}{\cosh \pi x_h} \, . \tag{2.32}$$

Summing up all the contributions (2.29), (2.30), (2.32), for the eigenvalue of the energy of the spin chain, we obtain

$$E = 2L\ln 2 - \sum_{h=1}^{H} \frac{\pi}{\cosh \pi x_h} +$$
$$+ \int_{-\infty}^{\infty} dy \left(\frac{d}{dy} \frac{1}{\cosh \pi y}\right) \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0)}\right] . \tag{2.33}$$

This expression is exact for any L and gives the largest anomalous dimensions of the gauge-invariant scalar operators of the SU(2) sub-sector in $\mathcal{N} = 4$ SYM. The first term on the r.h.s. is the known leading term proportional to L; the remaining two addends may be expanded in the limit $L \to \infty$ to provide respectively $\mathcal{O}(1)$ and $\mathcal{O}(1/L)$ corrections. Analytical expressions up to the order $1/L$ will be given in section 5.

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2.3 The momentum

The identification between the anomalous dimension of a gauge-invariant operator and the energy of a (spin chain) state needs to be supplemented by the zero momentum condition. Therefore, it is necessary to work out, by using the same technology as for the energy, the momentum eigenvalue[6]

$$P = \left(\sum_{k=1}^{M} p(u_k)\right) \mod 2\pi \,, \tag{2.34}$$

with the single particle momentum, defined as[7]

$$p(x) = \frac{1}{i}\ln\frac{x+\frac{i}{2}}{x-\frac{i}{2}} = \pi\,\mathrm{sign}(x) - 2\arctan 2x = \pi\,\mathrm{sign}(x) - \phi\left(x, \frac{1}{2}\right) . \tag{2.35}$$

This relation and the analogous (2.28) suggest the interpretation of each root as a particle (magnon) exciting the ferromagnetic vacuum and obeying the energy-momentum dispersion relation

$$e(u_k) = 4\sin^2\frac{p(u_k)}{2}\,, \qquad k = 1, \ldots, M.$$

We remark that $p(x)$ is odd and discontinuous in zero

$$p(x) + p(-x) = 0\,, \qquad \lim_{x\to 0^{\pm}} p(x) = \pm\pi\,.$$

The total momentum may be arranged so as to extract its non-analytic contribution

$$P = \pi(M_R^+ - M_R^-) - \sum_{k=1}^{M}\phi\left(u_k, \frac{1}{2}\right) = \pi M - \sum_{k=1}^{M}\phi\left(u_k, \frac{1}{2}\right) , \tag{2.36}$$

with M_R^+ the number of positive or zero real roots and M_R^- that of negative roots. The second equality is obtained, modulo 2π, by adding $2\pi M_R^-$. Now, we can apply formula (2.27) to the analytic part of $p(x)$, $p_{\mathrm{an}}(x) = -\phi\left(x, \frac{1}{2}\right)$. The first term vanishes in that the integrand is an odd function

$$-\int_{-\infty}^{\infty}\frac{dx}{2\pi}\,p'_{\mathrm{an}}(x)F(x) = 0\,. \tag{2.37}$$

For what concerns the contribution

$$\int_{-\infty}^{\infty}\frac{dx}{\pi}\,p'_{\mathrm{an}}(x)\int_{-\infty}^{\infty}dy\,[\delta(x-y) - G(x-y)]\,\mathrm{Im}\ln\left[1 + (-1)^{\delta}e^{iZ(y+i0)}\right] ,$$

we first evaluate the integration on x,

$$\int_{-\infty}^{\infty}dx\,p'_{\mathrm{an}}(x)[\delta(x-y) - G(x-y)] = -\int_{-\infty}^{\infty}dk\,e^{iky}\frac{e^{\frac{|k|}{2}}}{1+e^{|k|}}$$

[6]Of course, the identification up to 2π multiples comes from its definition as a displacement operator on a periodic (one-dimensional) lattice.

[7]We need to extend the definition of the sign-function so that $\mathrm{sign}(0) = 1$.

$$= -\int_{-\infty}^{\infty} dk\, \frac{e^{iky}}{2\cosh\frac{k}{2}} = -\frac{\pi}{\cosh \pi y}\,, \tag{2.38}$$

in order to obtain

$$\int_{-\infty}^{\infty} \frac{dx}{\pi}\, p'_{\rm an}(x) \int_{-\infty}^{\infty} dy\, [\delta(x-y) - G(x-y)]\, {\rm Im}\ln\left[1+(-1)^{\delta} e^{iZ(y+i0)}\right] =$$

$$= -\int_{-\infty}^{\infty} dy\, \frac{1}{\cosh \pi y}\, {\rm Im}\ln\left[1+(-1)^{\delta} e^{iZ(y+i0)}\right]\,. \tag{2.39}$$

Finally, we need to compute the term

$$\sum_{h=1}^{H} \left\{ \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, p'_{\rm an}(x)\, i \ln S(x-x_h) - p_{\rm an}(x_h) \right\}\,. \tag{2.40}$$

But this expression is the sum of single hole contributions which are minus the primitive of (2.38) at the value $y = x_h$. So, each term is given by

$$\int \frac{\pi}{\cosh \pi y} = \arctan \sinh \pi y + \text{const.}$$

And the integration constant is zero since the single term has to vanish for $y = x_h = 0$ for parity reasons ($p_{\rm an}$ is odd and G even). So (2.40) simplifies to

$$\sum_{h=1}^{H} \left\{ \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, p'_{\rm an}(x)\; i \ln S(x-x_h) - p_{\rm an}(x_h) \right\} = \sum_{h=1}^{H} \arctan\, \sinh \pi x_h\,. \tag{2.41}$$

All the contributions (2.37), (2.39), (2.41) yield the momentum eigenvalue

$$P = \pi M + \sum_{h=1}^{H} (\arctan \sinh \pi x_h) - \int_{-\infty}^{\infty} dy \frac{1}{\cosh \pi y}\, {\rm Im}\ln\left[1+(-1)^{\delta} e^{iZ(y+i0)}\right]\,. \tag{2.42}$$

The attentive reader will find similar results here and there in [31, 36].

As we said at the beginning of this subsection, the condition $P = 0$ works as a constraint for the anomalous dimension (2.33), (2.1). In particular, the antiferromagnetic state simply enjoys

$$P = \left(\pi \frac{L}{2}\right) \mod 2\pi = \begin{cases} 0 \text{ if } L \in 4\mathbb{N}\,, \\ \pi \text{ if } L \in 4\mathbb{N}+2\,, \end{cases} \tag{2.43}$$

so it possesses a SYM operator as a counterpart only if $L \in 4\mathbb{N}$.

3. The SU(2) case: multi-loops

Following the line of the previous section, we want to establish the NLIE framework for the conjectured multi-loop Bethe equations [7]

$$\left(\frac{X(u_j + \frac{i}{2})}{X(u_j - \frac{i}{2})} \right)^{L} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \tag{3.1}$$

where we introduced the function

$$X(x) = \frac{x}{2}\left(1 + \sqrt{1 - \frac{\lambda}{4\pi^2 x^2}}\right). \tag{3.2}$$

And as usual the single particle momentum $p(u_j, \lambda)$ is such that $e^{i\,p(u_j,\lambda)\,L}$ equals the l.h.s. of the corresponding Bethe equation (3.1), or explicitly

$$p(x, \lambda) = \frac{1}{i}\ln\frac{X(x + \frac{i}{2})}{X(x - \frac{i}{2})}. \tag{3.3}$$

Of course, these equations reproduce the Heisenberg case of section 2 in the small coupling limit $\lambda \to 0$: actually, only the l.h.s. in (3.1), namely the momentum (3.3), has changed from the XXX case. Therefore, in the present section we shall systematically follow all computations of the previous one.

In other words, we simply need to change the function $\phi(x, \frac{1}{2})$ of (2.4) into

$$\Phi(x, \lambda) = i\,\ln\frac{(\frac{i}{2} + x)\sqrt{1 - \frac{\lambda}{4\pi^2(x+\frac{i}{2})^2}}}{(\frac{i}{2} - x)\sqrt{1 - \frac{\lambda}{4\pi^2(x-\frac{i}{2})^2}}}, \tag{3.4}$$

which, contrarily to the momentum (3.3), is continuous in $x = 0$. Then, the counting function may be defined as

$$Z(x, \lambda) = L\,\Phi(x, \lambda) - \sum_{k=1}^{M}\phi(x - u_k, 1), \tag{3.5}$$

so that the Bethe equations read (with certain integer quantum numbers I_j)

$$\begin{aligned} Z(u_j, \lambda) &= \pi(2I_j + \delta - 1), \\ \delta &\equiv (L - M) \bmod 2. \end{aligned} \tag{3.6}$$

The asymptotic limits of the counting function are again given by

$$\lim_{x\to\pm\infty} Z(x, \lambda) = \pm(L - M)\,\pi$$

and can be used to fix the number of holes. Indeed, as $Z(x, \lambda)$ is an increasing function, the conditions (3.6) with generic integers I_j are at most satisfied by $L - M$ points on the real axis. This means that the number of holes is

$$H = L - 2M, \tag{3.7}$$

when considering states with real roots only. The position of any hole x_h is fixed by a quantisation condition identical to (3.6), but with a *fake* quantum number I_h

$$Z(x_h, \lambda) = \pi(2I_h + \delta - 1). \tag{3.8}$$

Consequently, both the Bethe roots and the holes, viz. $x = u_j, x_h$, satisfy the condition

$$\exp[iZ(x,\lambda)] = (-1)^{\delta-1} \,. \tag{3.9}$$

Again, as $\epsilon \to 0$ the sum over all the real roots (2.13) takes on the form

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) = & - \int_{-\infty}^{\infty} \frac{dy}{2\pi} f'(y)\, Z(y,\lambda) + \\ & + \int_{-\infty}^{\infty} \frac{dy}{\pi} f'(y) \,\mathrm{Im}\ln\left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] - \sum_{h=1}^{H} f(x_h)\,.\end{aligned} \tag{3.10}$$

It may be applied to the sum in the counting function (3.5) bringing

$$\begin{aligned}Z(x,\lambda) = & \, L\,\Phi(x,\lambda) - \int_{-\infty}^{\infty} \frac{dy}{2\pi} \phi'(x-y,1)\, Z(y,\lambda) + \\ & + \int_{-\infty}^{\infty} \frac{dy}{\pi} \phi'(x-y,1)\, \mathrm{Im}\ln\left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] + \\ & + \sum_{h=1}^{H} \phi(x - x_h, 1)\,.\end{aligned} \tag{3.11}$$

It is convenient to introduce the usual (cf. (2.16)) synthetic notation

$$L(x,\lambda) = \mathrm{Im}\ln\left[1 + (-1)^{\delta} e^{iZ(x+i0,\lambda)}\right] \,.$$

After x Fourier transforming all the terms and moving the first convolution to the l.h.s., we will obtain

$$\hat{Z}(k,\lambda) = \hat{F}(k,\lambda) + 2\hat{G}(k)\hat{L}(k,\lambda) + \sum_{h=1}^{H} e^{-ikx_h}\hat{H}(k)\,. \tag{3.12}$$

All terms are the same as before in the Heisenberg chain, except the forcing term that now depends on λ and whose Fourier transform reads

$$\hat{F}(k,\lambda) = L\frac{\hat{\Phi}(k,\lambda)}{1 + \frac{1}{2\pi}\hat{\phi}'(k,1)}\,. \tag{3.13}$$

The x Fourier transform of $\Phi'(x,\lambda)$ is given in terms of the Bessel function of the first kind J_0 [40]

$$\hat{\Phi}'(k,\lambda) = 2\pi\, e^{-\frac{|k|}{2}} J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)\,.$$

The series expansion $J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right) = (1 - \frac{k^2}{16\pi^2}\lambda + \mathcal{O}(\lambda^2))$ [40] shows clearly the change with respect to the Heisenberg chain. The function $\phi(x,1)$ is unchanged, so we can make use of (2.23) to arrive at the final expression

$$F(x,\lambda) = L\int_{-\infty}^{\infty} dk\, \frac{\sin kx\, J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{k\; 2\cosh\frac{k}{2}} = L\left(\mathrm{gd}\;\pi x - \frac{\lambda}{16}\frac{\sinh \pi x}{\cosh^2 \pi x} + \mathcal{O}(\lambda^2)\right)\,. \tag{3.14}$$

Inverting the Fourier transforms of (3.12) leads to the NLIE valid for this multi-loop Bethe equations[8]

$$\begin{aligned} Z(x,\lambda) = \; & F(x,\lambda) - i\sum_{h=1}^{H} \ln S(x-x_h) + \\ & +2\int_{-\infty}^{\infty} dy\; G(x-y)\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0,\lambda)}\right]\,. \end{aligned} \tag{3.15}$$

Of course, the convolution kernel G and the hole term S are the same as in section 2: what makes the difference is *simply* the different forcing term F. And besides the structure of this NLIE is quite the same as in many other models, except for the specific form of the above-computed functions: hence this similarity corroborates straight away the effectiveness of our method.

Therefore, we can still follow the result (2.27) on the Heisenberg chain, keeping in mind that here the forcing term is given by (3.14):

$$\begin{aligned} \sum_{k=1}^{M} f(u_k) = & -\int_{-\infty}^{\infty}\frac{dx}{2\pi}f'(x)F(x,\lambda) + \\ & +\int_{-\infty}^{\infty}\frac{dx}{\pi}f'(x)\int_{-\infty}^{\infty}dy\;[\delta(x-y)-G(x-y)]\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0,\lambda)}\right] + \\ & +\sum_{h=1}^{H}\left\{\int_{-\infty}^{\infty}\frac{dx}{2\pi}f'(x)\;i\ln S(x-x_h) - f(x_h)\right\}\,. \end{aligned} \tag{3.16}$$

3.1 Anomalous dimension

As typical in Bethe Ansatz theory, the energy of the spin chain, and thus the anomalous dimension in gauge theory, is given by a sum on all the Bethe roots

$$E = \sum_{j=1}^{M} e(u_j,\lambda)\,, \tag{3.17}$$

where the (even) single particle energy function equals

$$e(x,\lambda) = i\left\{\frac{1}{X(x+\frac{i}{2})} - \frac{1}{X(x-\frac{i}{2})}\right\}\,. \tag{3.18}$$

So, we just need to insert this function into (3.16)

$$\begin{aligned} \sum_{k=1}^{M} e(u_k,\lambda) = & -\int_{-\infty}^{\infty}\frac{dx}{2\pi}e'(x,\lambda)F(x,\lambda) + \\ & +\int_{-\infty}^{\infty}\frac{dx}{\pi}e'(x,\lambda)\int_{-\infty}^{\infty}dy[\delta(x-y)-G(x-y)]\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0,\lambda)}\right] + \\ & +\sum_{h=1}^{H}\left\{\int_{-\infty}^{\infty}\frac{dx}{2\pi}e'(x,\lambda)\;i\ln S(x-x_h) - e(x_h,\lambda)\right\}\,, \end{aligned} \tag{3.19}$$

[8]We need to stress anew the absence of an integrable model behind them.

and re-call its Fourier transform

$$\hat{e}(k,\lambda) = \frac{8\pi^2 J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\sqrt{\lambda}\; k\; e^{\frac{|k|}{2}}}\,.$$

In fact, the first contribution reads

$$\begin{aligned}
-\int_{-\infty}^{\infty}\frac{dx}{2\pi}e'(x,\lambda)\,F(x,\lambda) &= \int_{-\infty}^{\infty}\frac{dx}{2\pi}\;e(x,\lambda)\,F'(x,\lambda)\\
&= \frac{8\pi}{\sqrt{\lambda}}L\int_0^{\infty}dk\;\frac{J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{k\,(e^k+1)}\\
&= L\left(2\ln 2-\frac{9\zeta(3)}{8(2\pi)^2}\lambda+\mathcal{O}(\lambda^2)\right)\,. \qquad (3.20)
\end{aligned}$$

Of course, it is the leading term of the anti-ferromagnetic state energy and thus coincides with the expression (10) of [16] (or (17,18) of [41]), where it was interestingly identified with the ground state energy of the half-filled Hubbard model. Moreover, the second contribution in (3.19) may be re-organised by expressing the convolution as an ordinary product in the Fourier space:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{dx}{\pi}\,e'(x,\lambda)\int_{-\infty}^{\infty}dy\,[\delta(x-y)-G(x-y)]\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0,\lambda)}\right]=\\
&\qquad = \int_{-\infty}^{\infty}dy\,e_1(y,\lambda)\,\mathrm{Im}\ln\left[1+(-1)^{\delta}e^{iZ(y+i0,\lambda)}\right]\,, \qquad (3.21)
\end{aligned}$$

where we made use of a new function as a Fourier transform of a construct of the Bessel function J_1

$$\begin{aligned}
e_1(x,\lambda) &= \int_{-\infty}^{\infty}dk\,\frac{2iJ_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)e^{ikx}}{\sqrt{\lambda}\,\cosh\frac{k}{2}} \qquad (3.22)\\
&= \frac{d}{dx}\left[\frac{1}{\cosh\pi x}+\frac{\lambda}{32\pi^2}\frac{d^2}{dx^2}\frac{1}{\cosh\pi x}+\mathcal{O}(\lambda^2)\right]\,.
\end{aligned}$$

The single hole contribution in (3.19) is now evaluated by integrating by parts and computing the convolution in the Fourier space:

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dx}{2\pi}e'(x,\lambda)i\ln S(x-x_h)-e(x_h,\lambda) &= \int_{-\infty}^{\infty}dx e(x,\lambda)[G(x-x_h)-\delta(x-x_h)]\\
&= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\;e^{ikx_h}\,\hat{e}(k,\lambda)\;[\hat{G}(k)-1]\\
&= -\int_{-\infty}^{\infty}dk\,2\pi\,e^{ikx_h}\,\frac{J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\sqrt{\lambda}\;k\cosh\frac{k}{2}} \qquad (3.23)\\
&= -\frac{\pi}{\cosh\pi x_h}-\frac{\lambda}{32\pi^2}\left[\frac{d^2}{dx^2}\frac{\pi}{\cosh\pi x}\right]_{x=x_h}+\mathcal{O}(\lambda^2)\,.
\end{aligned}$$

Eventually, we collect all terms (3.20), (3.21), (3.23) that form the energy (3.17) and obtain

$$\begin{aligned}
E = {} & \frac{8\pi}{\sqrt{\lambda}} L \int_0^\infty dk\, \frac{J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right) J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{k(e^k+1)} - \\
& - \sum_{h=1}^{H} \int_{-\infty}^{\infty} dk\; 2\pi\, e^{ikx_h} \frac{J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\sqrt{\lambda}\, k \cosh\frac{k}{2}} + \\
& + \int_{-\infty}^{\infty} dy \left(\int_{-\infty}^{\infty} dk\, \frac{2i J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right) e^{iky}}{\sqrt{\lambda}\ \cosh\frac{k}{2}} \right) \operatorname{Im}\ln\left[1+(-1)^\delta e^{iZ(y+i0,\lambda)}\right] .
\end{aligned} \tag{3.24}$$

This expression for the anomalous dimension is exact in any regime of L and specifically the second and third terms provide all the sub-leading corrections when $L \to \infty$, whose expressions are by the same token novel and intriguing. Analytical expressions of them up to the order $1/L$ will be given in section 5.

In conclusion, it is worth emphasising that the break-down of the proposal (3.1) at order g^{2L} affects our results as a trivial consequence, although it reveals itself unrelated to our method. In other words, the latter should be perfectly applicable to the hypothetic correct Bethe Ansatz equations, provided in the *typical* form, along similar steps.

3.2 Momentum

We can compute the momentum (2.34) by summing the single particle momenta (3.3) of all the Bethe roots. From (3.3) we separate the analytic contribution $p_{\rm an}(u_k,\lambda) = -\Phi(u_k,\lambda)$ so that, as in (2.36), we can write

$$P = \pi M - \sum_{k=1}^{M} \Phi(u_k,\lambda)\,. \tag{3.25}$$

Thus, we only need to apply (3.16) to the function $p_{\rm an}(x,\lambda)$. The first contribution vanishes

$$-\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, p'_{\rm an}(x,\lambda)\, F(x,\lambda) = 0 \tag{3.26}$$

as the integrand is the product of an even and an odd function. The second contribution to (3.16) becomes easily

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{\pi}\, p'_{\rm an}(x,\lambda) \int_{-\infty}^{\infty} dy\, [\delta(x-y) - G(x-y)] \operatorname{Im}\ln\left[1+(-1)^\delta e^{iZ(y+i0,\lambda)}\right] = \\
= -\int_{-\infty}^{\infty} \frac{dy}{\pi} \int_{-\infty}^{\infty} dk\, \frac{e^{iky} J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{2\cosh\frac{k}{2}} \operatorname{Im}\ln\left[1+(-1)^\delta e^{iZ(y+i0,\lambda)}\right] \\
= -\int_{-\infty}^{\infty} \frac{dy}{\pi}\, \frac{F'(y,\lambda)}{L} \operatorname{Im}\ln\left[1+(-1)^\delta e^{iZ(y+i0,\lambda)}\right] ,
\end{aligned} \tag{3.27}$$

with the appearance of the forcing/momentum term (3.14). The latter also appears in the hole contribution to (3.16)

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} p'_{\rm an}(x,\lambda)\, i \ln S(x-x_h) - p(x_h,\lambda) =$$
$$= \int_{-\infty}^{\infty} dx\, p_{\rm an}(x,\lambda) \left[G(x-x_h) - \delta(x-x_h)\right] = \frac{F(x_h,\lambda)}{L} . \quad (3.28)$$

In summary, we have the analogue of (2.42) in the present case

$$P = \pi M + \frac{1}{L}\sum_{h=1}^{H} F(x_h,\lambda) - \int_{-\infty}^{\infty} \frac{dy}{\pi} \frac{F'(y,\lambda)}{L} \,{\rm Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] . \quad (3.29)$$

This completes the general results of the multi-loop scenario and will allow us to extract in section 5 the first finite size corrections analytically (and explicitly).

4. The SO(6) scalar sector at one loop: finite size results

We want to illustrate the utility of the NLIE to compute the exact finite size contributions to the anomalous dimensions and momenta in the SO(6) scalar sector at one loop. Therefore, we need to consider a chain of L six-dimensional vectors of the $so(6)$ representation. As well known after [42], the Bethe Ansatz diagonalization of all the commuting integrals of motion is founded on the following system of coupled Bethe equations:

$$\begin{aligned}
\left(\frac{u_{1,j}+i/2}{u_{1,j}-i/2}\right)^L &= \prod_{\substack{k=1\\k\neq j}}^{M_1} \frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i} \prod_{k=1}^{M_2} \frac{u_{1,j}-u_{2,k}-i/2}{u_{1,j}-u_{2,k}+i/2} \prod_{k=1}^{M_3} \frac{u_{1,j}-u_{3,k}-i/2}{u_{1,j}-u_{3,k}+i/2} , \\
1 &= \prod_{\substack{k=1\\k\neq j}}^{M_2} \frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i} \prod_{k=1}^{M_1} \frac{u_{2,j}-u_{1,k}-i/2}{u_{2,j}-u_{1,k}+i/2} , \qquad (4.1)\\
1 &= \prod_{\substack{k=1\\k\neq j}}^{M_3} \frac{u_{3,j}-u_{3,k}+i}{u_{3,j}-u_{3,k}-i} \prod_{k=1}^{M_1} \frac{u_{3,j}-u_{1,k}-i/2}{u_{3,j}-u_{1,k}+i/2} .
\end{aligned}$$

By making use of the function (2.5), we may define three counting functions, i.e. one for each group of Bethe equations,

$$\begin{aligned}
Z_1(u) &= L\,\phi(u,1/2) - \sum_{k=1}^{M_1} \phi(u-u_{1,k},1) + \\
&\quad + \sum_{k=1}^{M_2} \phi(u-u_{2,k},1/2) + \sum_{k=1}^{M_3} \phi(u-u_{3,k},1/2) , \\
Z_2(u) &= -\sum_{k=1}^{M_2} \phi(u-u_{2,k},1) + \sum_{k=1}^{M_1} \phi(u-u_{1,k},1/2) , \qquad (4.2)\\
Z_3(u) &= -\sum_{k=1}^{M_3} \phi(u-u_{3,k},1) + \sum_{k=1}^{M_1} \phi(u-u_{1,k},1/2) ,
\end{aligned}$$

such that the Bethe equations look as

$$
\begin{aligned}
Z_1(u_{1,j}) &= \pi(2I_{1,j} + \delta_1 - 1)\,, \\
Z_2(u_{2,j}) &= \pi(2I_{2,j} + \delta_2 - 1)\,, \\
Z_3(u_{3,j}) &= \pi(2I_{3,j} + \delta_3 - 1)\,,
\end{aligned} \tag{4.3}
$$

where $I_{k,i}$ are integer quantum numbers and the (2.6) has been generalised to

$$
\begin{aligned}
\delta_1 &= (L - M_1 + M_2 + M_3) \bmod 2\,, \\
\delta_2 &= (M_1 - M_2) \bmod 2\,, \\
\delta_3 &= (M_1 - M_3) \bmod 2\,.
\end{aligned} \tag{4.4}
$$

We now consider states of the chain described by real solutions $\{u_{k,i}\}$ to the Bethe equations. For simplicity's sake, we consider the case in which the parities of the integers L, M_1, M_2, M_3 are such that $\exp[iZ_k(u_{k,i})] = -1$ (compare with (2.12)). In addition, even if the formalism would allow us to consider states with holes (and generally complex roots), here we limit ourselves to states which contain no holes, i.e. such that the points u satisfying $\exp[iZ_k(u)] = -1$ are exhausted by a (real) solution set $\{u_{k,i}\}$: we will be extending our results in an incoming publication [43]. As before, these requirements will constrain the allowed values of M_k. Indeed, from the limits (2.9) we obtain the limiting values

$$
\begin{aligned}
\lim_{x\to\pm\infty} Z_1(x) &= \pm\pi(L - M_1 + M_2 + M_3)\,, \\
\lim_{x\to\pm\infty} Z_2(x) &= \pm\pi(M_1 - M_2)\,, \\
\lim_{x\to\pm\infty} Z_3(x) &= \pm\pi(M_1 - M_3)\,.
\end{aligned}
$$

Imposing the condition that the points u satisfying $\exp[iZ_k(u)] = -1$ are those and only those in the solution set $\{u_{k,i}\}$ furnishes these constraints

$$
|L - M_1 + M_2 + M_3| = M_1\,, \quad |M_1 - M_2| = M_2\,, \quad |M_1 - M_3| = M_3\,, \tag{4.5}
$$

whose solution, if $M_1 \neq 0$, is

$$
M_1 = L\,, \quad M_2 = M_3 = \frac{L}{2}\,. \tag{4.6}
$$

If $L \in 4\mathbb{N}$ there is one single state with these features and it is the completely anti-ferromagnetic state: as discussed in [3] it is the state with maximal energy and zero momentum, too. Now, the usual procedure allows us to write a sum over the Bethe roots of a function f (analytic within a strip around the real axis) in terms of integrals involving the Zs:

$$
\sum_{i=1}^{M_k} f(u_{k,i}) = -\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, f'(x) Z_k(x) + \int_{-\infty}^{\infty} \frac{dx}{\pi}\, f'(x)\, \mathrm{Im}\ln\left[1 + e^{iZ_k(x+i0)}\right]\,. \tag{4.7}
$$

Upon applying (4.7) in the definition of Z_k, we are on the road to write NLIEs for the counting functions:

$$\begin{aligned}
Z_1(x) &= L\,\phi(x,1/2) - \int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,Z_1(y) + \\
&\quad +2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,\mathrm{Im}\ln\left[1+e^{iZ_1(y+i0)}\right] + \\
&\quad + \int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,Z_2(y) - 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,\mathrm{Im}\ln\left[1+e^{iZ_2(y+i0)}\right] + \\
&\quad + \int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,Z_3(y) - 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,\mathrm{Im}\ln\left[1+e^{iZ_3(y+i0)}\right], \\
Z_2(x) &= -\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,Z_2(y) + 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,\mathrm{Im}\ln\left[1+e^{iZ_2(y+i0)}\right] + \\
&\quad + \int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,Z_1(y) - 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,\mathrm{Im}\ln\left[1+e^{iZ_1(y+i0)}\right], \\
Z_3(x) &= -\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,Z_3(y) + 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1)\,\mathrm{Im}\ln\left[1+e^{iZ_3(y+i0)}\right] + \\
&\quad + \int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,Z_1(y) - 2\int_{-\infty}^{\infty} \frac{dy}{2\pi}\,\phi'(x-y,1/2)\,\mathrm{Im}\ln\left[1+e^{iZ_1(y+i0)}\right].
\end{aligned}$$

By symmetry considerations (note that $M_2 = M_3 = L/2$) we can infer that $Z_2(x) = Z_3(x)$, so that we have to deal with only two equations. We put again Fourier transforms into the game and obtain

$$\begin{aligned}
\hat{Z}_1(k) &= L\,\hat{\phi}(k,1/2) - \frac{1}{2\pi}\,\hat{\phi}'(k,1)\,\hat{Z}_1(k) + 2\,\frac{1}{2\pi}\,\hat{\phi}'(k,1)\,\hat{L}_1(k) + \\
&\quad +2\,\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)\,\hat{Z}_2(k) - 4\,\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)\,\hat{L}_2(k), \\
\hat{Z}_2(k) &= -\frac{1}{2\pi}\,\hat{\phi}'(k,1)\,\hat{Z}_2(k) + 2\,\frac{1}{2\pi}\,\hat{\phi}'(k,1)\,\hat{L}_2(k) + \\
&\quad +\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)\,\hat{Z}_1(k) - 2\,\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)\,\hat{L}_1(k)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\hat{Z}_1(k) &= \frac{L\,\hat{\phi}(k,1/2)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)} + 2\frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{L}_1(k) + \\
&\quad +2\frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{Z}_2(k) - 4\frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{L}_2(k), \\
\hat{Z}_2(k) &= 2\frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{L}_2(k) + \frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{Z}_1(k) - \\
&\quad -2\frac{\frac{1}{2\pi}\,\hat{\phi}'(k,1/2)}{1+\frac{1}{2\pi}\,\hat{\phi}'(k,1)}\hat{L}_1(k).
\end{aligned}$$

No need to say that the usual short notation

$$\hat{L}_i(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \, \mathrm{Im} \ln \left[1 + e^{iZ_i(x+i0)} \right] , \quad i = 1, 2 \, ,$$

has appeared. Thus, clearly the x Fourier transform of the ubiquitous function $\phi(x, \xi)$,

$$\hat{\phi}(k, \xi) = -2\pi i e^{-\xi|k|} P \left(\frac{1}{k} \right) , \tag{4.8}$$

plays a central rôle to achieve

$$\begin{aligned}
\hat{Z}_1(k) &= -L \, 2\pi i \, \frac{e^{-\frac{|k|}{2}}}{1+e^{-|k|}} P \left(\frac{1}{k} \right) + 2 \frac{e^{-|k|}}{1+e^{-|k|}} \, \hat{L}_1(k) + \\
&\quad + 2 \frac{e^{-\frac{|k|}{2}}}{1+e^{-|k|}} \, \hat{Z}_2(k) - 4 \frac{e^{-\frac{|k|}{2}}}{1+e^{-|k|}} \, \hat{L}_2(k) \, , \\
\hat{Z}_2(k) &= 2 \frac{e^{-|k|}}{1+e^{-|k|}} \, \hat{L}_2(k) + \frac{e^{-\frac{|k|}{2}}}{1+e^{-|k|}} \, \hat{Z}_1(k) - 2 \frac{e^{-\frac{|k|}{2}}}{1+e^{-|k|}} \, \hat{L}_1(k) \, .
\end{aligned}$$

Eventually, these equations can be re-arranged in the clearer manner

$$\begin{aligned}
\hat{Z}_1(k) &= -L \, 2\pi i \, e^{-\frac{|k|}{2}} \frac{1+e^{-|k|}}{1+e^{-2|k|}} P \left(\frac{1}{k} \right) - 2e^{-|k|} \frac{1-e^{-|k|}}{1+e^{-2|k|}} \, \hat{L}_1(k) - 4 \frac{e^{-\frac{|k|}{2}}}{1+e^{-2|k|}} \, \hat{L}_2(k) \, , \\
\hat{Z}_2(k) &= -L \, 2\pi i \, \frac{e^{-|k|}}{1+e^{-2|k|}} P \left(\frac{1}{k} \right) - 2 \frac{e^{-\frac{|k|}{2}}}{1+e^{-2|k|}} \, \hat{L}_1(k) - 2e^{-|k|} \frac{1-e^{-|k|}}{1+e^{-2|k|}} \, \hat{L}_2(k) \, ,
\end{aligned}$$

or, after anti-transforming, in the final NLIEs for the Zs

$$\begin{aligned}
Z_1(x) &= F_1(x) + 2 \int_{-\infty}^{\infty} dy \, G_{11}(x-y) \, \mathrm{Im} \ln \left[1 + e^{iZ_1(y+i0)} \right] + \\
&\quad + 2 \int_{-\infty}^{\infty} dy \, G_{12}(x-y) \, \mathrm{Im} \ln \left[1 + e^{iZ_2(y+i0)} \right] , \qquad (4.9) \\
Z_2(x) &= F_2(x) + 2 \int_{-\infty}^{\infty} dy \, G_{21}(x-y) \, \mathrm{Im} \ln \left[1 + e^{iZ_1(y+i0)} \right] + \\
&\quad + 2 \int_{-\infty}^{\infty} dy \, G_{22}(x-y) \, \mathrm{Im} \ln \left[1 + e^{iZ_2(y+i0)} \right] . \qquad (4.10)
\end{aligned}$$

The known functions in the previous equations are explicitly

$$\begin{aligned}
F_1(x) &= 2L \int_0^{\infty} dk \, \frac{\sin kx}{k} \, \frac{\cosh \frac{k}{2}}{\cosh k} = 2L \, \arctan \left(\sqrt{2} \sinh \frac{\pi x}{2} \right) , \\
F_2(x) &= L \int_0^{\infty} \frac{dk}{k} \, \frac{\sin kx}{\cosh k} = L \, \mathrm{gd} \frac{\pi x}{2} \, , \\
G_{11}(x) &= G_{22}(x) = - \int_0^{\infty} \frac{dk}{2\pi} \, \cos kx \, \frac{1-e^{-k}}{\cosh k} \qquad (4.11) \\
&= -\frac{1}{4} \frac{1}{\cosh \frac{\pi x}{2}} + \frac{1}{2\pi i} \frac{d}{dx} \ln \frac{\Gamma\left(1+\frac{ix}{4}\right) \Gamma\left(\frac{1}{2}-\frac{ix}{4}\right)}{\Gamma\left(1-\frac{ix}{4}\right) \Gamma\left(\frac{1}{2}+\frac{ix}{4}\right)} \, ,
\end{aligned}$$

$$
\begin{aligned}
G_{12}(x) &= 2\,G_{21}(x) = -\int_0^\infty \frac{dk}{\pi}\,\cos kx\,\frac{e^{\frac{k}{2}}}{\cosh k} \\
&= -\frac{1}{\sqrt{2}}\frac{\cosh\frac{\pi x}{2}}{\cosh \pi x} - \frac{1}{2}\frac{1}{\cosh \pi x} + \frac{1}{\pi i}\frac{d}{dx}\ln\frac{\Gamma\left(\frac{7}{8}+\frac{ix}{4}\right)\Gamma\left(\frac{5}{8}-\frac{ix}{4}\right)}{\Gamma\left(\frac{7}{8}-\frac{ix}{4}\right)\Gamma\left(\frac{5}{8}+\frac{ix}{4}\right)}\,.
\end{aligned}
$$

Summarizing, the equations (4.9), (4.10) are the Non-Linear Integral Equations describing the anti-ferromagnetic state (real solutions without holes to the Bethe equations) of the SO(6) symmetric chain (vector representation). This state possesses zero momentum and the maximal energy. The latter will receive an exact expression — in terms of solutions of the NLIEs (4.9), (4.10) — in the next subsection.

4.1 The anomalous dimension

The important result of [3] is that the dilatation matrix of scalar operators in $\mathcal{N}=4$ SYM at one loop can be mapped to the hamiltonian of an integrable SO(6) symmetric chain. In terms of the Bethe roots, its eigenvalue γ reads as follows:

$$
\gamma = \frac{\lambda}{16\pi^2}E\,, \quad E = 2\sum_{i=1}^{M_1}\frac{1}{u_{1,i}^2+\frac{1}{4}}\,, \tag{4.12}
$$

where E is the chain energy. The maximal eigenvalue (anomalous dimension) is obtained when considering the solution to the Bethe equations containing real roots and no holes. For this configuration, by the same arguments used in the previous sections, a sum over the set 1 of Bethe roots can be expressed in terms of integrals involving Z_1:

$$
\begin{aligned}
\sum_{i=1}^{M_1} f(u_{1,i}) &= -\int_{-\infty}^{\infty}\frac{dx}{2\pi}\,f'(x)\,Z_1(x) + 2\int_{-\infty}^{\infty}\frac{dx}{2\pi}\,f'(x)\,\mathrm{Im}\ln\left[1+e^{iZ_1(x+i0)}\right] = \\
&= -\int_{-\infty}^{\infty}\frac{dk}{(2\pi)^2}\,\hat{f}'(k)\,\hat{Z}_1(-k) + 2\int_{-\infty}^{\infty}\frac{dk}{(2\pi)^2}\,\hat{f}'(k)\,\hat{L}_1(-k)\,.
\end{aligned}
$$

Inserting now the NLIE (4.9) for Z_1 yields

$$
\begin{aligned}
\sum_{i=1}^{M_1} f(u_{1,i}) &= L\int_{-\infty}^{\infty}\frac{dk}{2\pi i}\,\hat{f}'(k)\,e^{-\frac{|k|}{2}}\,\frac{1+e^{-|k|}}{1+e^{-2|k|}}\,P\left(\frac{1}{k}\right) + \\
&+ \int_{-\infty}^{\infty}\frac{dk}{2\pi^2}\,\hat{f}'(k)\,\frac{1+e^{-|k|}}{1+e^{-2|k|}}\,\hat{L}_1(-k) + \int_{-\infty}^{\infty}\frac{dk}{4\pi^2}\,\hat{f}'(k)\,4\,\frac{e^{-\frac{|k|}{2}}}{1+e^{-2|k|}}\,\hat{L}_2(-k)\,.
\end{aligned}
$$

Upon specializing f to be the single particle energy

$$
E = 2\sum_{i=1}^{M_1}\frac{1}{u_{1,i}^2+\frac{1}{4}} \Rightarrow f(u) = e(u) = \frac{2}{u^2+\frac{1}{4}}\,, \quad \hat{e'}(k) = 4\pi i k\,e^{-\frac{|k|}{2}}\,, \tag{4.13}
$$

we obtain

$$
\begin{aligned}
E &= 4L\int_0^\infty dk\,e^{-k}\,\frac{1+e^{-k}}{1+e^{-2k}} + 2\frac{i}{\pi}\int_{-\infty}^{\infty}dk\,k\,\frac{\cosh\frac{k}{2}}{\cosh k}\,\hat{L}_1(-k) + \\
&+ 2\frac{i}{\pi}\int_{-\infty}^{\infty}dk\,k\,\frac{1}{\cosh k}\hat{L}_2(-k)\,.
\end{aligned} \tag{4.14}
$$

The first term in the r.h.s. of (4.14) gives the leading contribution when $L \to \infty$. It may be easily evaluated

$$4L \int_0^\infty dk\, e^{-k} \frac{1+e^{-k}}{1+e^{-2k}} = 4L \int_0^1 dx\, \frac{1+x}{1+x^2} = 2L \left(\frac{\pi}{2} + \ln 2 \right) . \tag{4.15}$$

We remark that (4.15) agrees with (5.10) of [3]. Hence, the maximal energy may be re-written as

$$\begin{aligned} E &= 2L \left(\frac{\pi}{2} + \ln 2 \right) + \int_{-\infty}^{\infty} dx\, E_1(x)\, \mathrm{Im} \ln \left[1 + e^{iZ_1(x+i0)} \right] + \\ &+ \int_{-\infty}^{\infty} dx\, E_2(x)\, \mathrm{Im} \ln \left[1 + e^{iZ_2(x+i0)} \right] , \end{aligned} \tag{4.16}$$

where the last two terms contain the functions

$$E_1(x) = 2\sqrt{2}\, \frac{d}{dx} \frac{\cosh \frac{\pi x}{2}}{\cosh \pi x} , \quad E_2(x) = -\pi\, \frac{\sinh \frac{\pi x}{2}}{\cosh^2 \frac{\pi x}{2}} . \tag{4.17}$$

Formula (4.16) is an exact expression for the energy in terms of the solution of the NLIEs (4.9), (4.10). When $L \to \infty$, the last two terms provide the $\mathcal{O}(1/L)$ corrections to the anomalous dimension. Analytical computations up to the order $1/L$ will be the topic of the next section.

5. Analytic calculation of sub-leading order

It turns out that in all the discussed cases it is even possible to single out the explicit sub-leading contribution to the energy as $L \to \infty$. In fact, it is of order $1/L$ and comes out in a rather standard way by following the strategy of the 'derivative lemma' [32].

Also, we need to mention that higher order corrections might be extracted explicitly in the framework of NLIEs; for Klümper was able to compute the first logarithmic corrections (to the $1/L$ term) in the spin 1/2-XXX Heisenberg chain starting by a NLIE similar to ours, although using some numerical insights [36] (see also [44] and references therein for a comparison with computations in a field theoretic framework). For the time being, we are not interested in discussing the analytic derivation of these contributions, but will return to them in a subsequent paper [43]. Nonetheless, their presence too is expected in the multi-loop SU(2) and SO(6) cases, as motivated in section 6, where we will present our numerical results.

XXX Heisenberg chain. Let us first consider the Heisenberg chain and the excitations over the anti-ferromagnetic state described by holes. The contributions we want to evaluate are contained in the integration term of (2.33):

$$\begin{aligned} \Delta E(L) &= \int_{-\infty}^{\infty} dy\, (-\pi)\, \frac{\sinh \pi y}{\cosh^2 \pi y}\, \mathrm{Im} \ln \left[1 + (-1)^\delta e^{iZ(y+i0)} \right] \\ &= \int_{-\infty}^{0} dy\, (-\pi)\, \frac{\sinh \pi y}{\cosh^2 \pi y}\, \mathrm{Im} \ln \left[1 + (-1)^\delta e^{iZ(y+i0)} \right] + \\ &+ \int_{0}^{\infty} dy\, (-\pi)\, \frac{\sinh \pi y}{\cosh^2 \pi y}\, \mathrm{Im} \ln \left[1 + (-1)^\delta e^{iZ(y+i0)} \right] . \end{aligned} \tag{5.1}$$

In order to single out the order $1/L$ contributions, we perform different changes of variables in each of the integrals in (5.1),

$$y = \theta - \frac{\ln 2L}{\pi}\,, \quad \text{for } y < 0\,, \quad y = \theta + \frac{\ln 2L}{\pi}\,, \quad \text{for } y > 0\,,$$

and then we let $L \to \infty$. In this limit we have

$$\begin{aligned}\Delta E(L) \;=\; & \frac{\pi}{L}\int_{-\infty}^{\infty} d\theta \left\{ e^{\pi\theta}\,\mathrm{Im}\ln\left[1+e^{iZ^-(\theta+i0)}\right] - e^{-\pi\theta}\,\mathrm{Im}\ln\left[1+e^{iZ^+(\theta+i0)}\right]\right\} + \\ & + o(1/L)\,, \end{aligned} \tag{5.2}$$

where the usual symbol $o(1/L)$ is used to indicate terms that vanish faster than $1/L$

$$\lim_{L\to\infty} o(1/L)\, L = 0\,. \tag{5.3}$$

The new functions

$$Z^{\mp}(\theta) = \lim_{L\to\infty}\left[Z\left(\theta \mp \frac{\ln 2L}{\pi}\right) \pm \frac{\pi}{2}(H+L)\right]\,,$$

satisfy the *kink* NLIEs:

$$\begin{aligned} Z^-(\theta) \;&=\; e^{\pi\theta} + 2\int_{-\infty}^{\infty} d\theta'\; G(\theta-\theta')\;\mathrm{Im}\ln\left[1+e^{iZ^-(\theta'+i0)}\right]\,, \\ Z^+(\theta) \;&=\; -e^{-\pi\theta} + 2\int_{-\infty}^{\infty} d\theta'\; G(\theta-\theta')\,\mathrm{Im}\ln\left[1+e^{iZ^+(\theta'+i0)}\right]\,. \end{aligned} \tag{5.4}$$

We notice that the holes contribution in the NLIE (2.25) has become a constant and has been reabsorbed in a redefinition of the counting function.

In summary, the integral term in (5.2) gives the contribution of order $1/L$ to the energy (as $L \to \infty$). This term can be exactly computed by using the derivative lemma based on the dilogarithmic function (for an enunciation of this lemma see for instance section 7 of [32]). The consequent result for $\Delta E(L)$ is

$$\Delta E(L) = \frac{\pi^2}{6L} + o(1/L) \tag{5.5}$$

and it does not depend on the holes, as long as their position remains finite for large L.

We can now evaluate the holes contribution to the energy (2.33). From (2.11) we deduce the leading behaviour

$$x_h \sim \frac{2I_h + \delta - 1}{L}\,, \tag{5.6}$$

namely holes accumulate towards the point $x = 0$. This leads to the following contribution,

$$-\sum_{h=1}^{H} \frac{\pi}{\cosh \pi x_h} = -H\,\pi + \frac{\pi^3}{2L^2}\sum_{h=1}^{H}(2I_h+\delta-1)^2 + \mathcal{O}(1/L^3)\,, \tag{5.7}$$

from which we conclude that hole excitations over the anti-ferromagnetic state do not contribute to $1/L$ terms. In summary, the energy is given by

$$E = 2L\ln 2 - \pi H + \frac{\pi^2}{6L} + o(1/L)\,, \tag{5.8}$$

where the symbol $o(1/L)$ is defined in (5.3).

The SU(2) **multi-loop chain.** In this case we will evaluate explicitly the energy (3.24) up to the order $1/L$ (in the limit $L \to \infty$). Let us start by the term,

$$\Delta E(L,\lambda) = \int_{-\infty}^{\infty} dy\ e_1(y,\lambda)\, \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] , \tag{5.9}$$

where $e_1(x,\lambda)$ has been defined in (3.22). As before, we split the integral (5.9) into two parts,

$$\begin{aligned}\Delta E(L,\lambda) \ =\ & \int_{-\infty}^{0} dy\ e_1(y,\lambda)\ \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] + \\ & + \int_{0}^{\infty} dy\ e_1(y,\lambda)\, \mathrm{Im} \ln \left[1 + (-1)^{\delta} e^{iZ(y+i0,\lambda)}\right] ,\end{aligned}$$

we perform the change of variables,

$$y = \theta - \frac{\ln 2L}{\pi} , \quad \text{for } y < 0 , \qquad y = \theta + \frac{\ln 2L}{\pi} , \quad \text{for } y > 0 , \tag{5.10}$$

and then we let $L \to \infty$. In this limit the two integrals can be computed by using the residue method. The first (second) one is evaluated after choosing a contour closing in the lower (upper) complex y-plane. The poles of the integrand lie on the imaginary axis, $k = \pm i\pi(2n+1)$, $n \geq 0$ and give contributions proportional to L^{-2n-1}. Restricting to the leading $1/L$ contribution in the limit $L \to \infty$, we can write

$$\begin{aligned}\Delta E(L,\lambda) \ =\ & \frac{4\pi i}{L\sqrt{\lambda}} J_1\left(-i\frac{\sqrt{\lambda}}{2}\right) \int_{-\infty}^{\infty} d\theta\, e^{\pi\theta}\, \mathrm{Im} \ln \left[1 + e^{iZ^-(\theta+i0,\lambda)}\right] - \\ & - \frac{4\pi i}{L\sqrt{\lambda}} J_1\left(-i\frac{\sqrt{\lambda}}{2}\right) \int_{-\infty}^{\infty} d\theta\, e^{-\pi\theta}\, \mathrm{Im} \ln \left[1 + e^{iZ^+(\theta+i0,\lambda)}\right] + o(1/L) ,\end{aligned} \tag{5.11}$$

where $o(1/L)$ satisfies (5.3) and, as before, we have defined the new functions

$$Z^{\mp}(\theta,\lambda) = \lim_{L\to\infty} \left[Z\left(\theta \mp \frac{\ln 2L}{\pi}, \lambda\right) \pm \frac{\pi}{2}(H+L)\right] . \tag{5.12}$$

The equations satisfied by $Z^{\mp}(\theta,\lambda)$ are obtained starting from (3.15), performing the shifts appearing in their definition (5.12), then evaluating the leading contribution of holes and the forcing term when $L \to \infty$. The holes term gives, as before, $\mp\pi H/2$. On the other hand, the forcing term can be evaluated by using the residue technique in a fashion similar to the previous energy kernel calculation. Its contribution is

$$\mp\frac{\pi L}{2} \pm J_0\left(i\frac{\sqrt{\lambda}}{2}\right) e^{\pm\pi\theta} + \mathcal{O}(1/L) , \tag{5.13}$$

the first term coming from the residue in $k = 0$, the second from the residues in $k = \pm i\pi$ of the integrand of (3.14). It follows that the equations satisfied by $Z^{\mp}(\theta,\lambda)$ take the form

$$Z^-(\theta,\lambda) = J_0\left(i\frac{\sqrt{\lambda}}{2}\right) e^{\pi\theta} + 2\int_{-\infty}^{\infty} d\theta'\ G(\theta-\theta')\ \mathrm{Im} \ln \left[1 + e^{iZ^-(\theta'+i0,\lambda)}\right] , \tag{5.14}$$

$$Z^+(\theta,\lambda) = -J_0\left(i\frac{\sqrt{\lambda}}{2}\right)e^{-\pi\theta} + 2\int_{-\infty}^{\infty} d\theta'\; G(\theta-\theta')\,\mathrm{Im}\ln\left[1+e^{iZ^+(\theta'+i0,\lambda)}\right].$$

We remark that we are still able to apply the derivative lemma in order to compute the $1/L$ contributions of (5.11), the only difference with the Heisenberg chain being that now the functions involved are dressed with Bessel functions. The result is

$$\Delta E(L,\lambda) = \frac{4i}{L\sqrt{\lambda}}\frac{J_1\left(-i\frac{\sqrt{\lambda}}{2}\right)}{J_0\left(i\frac{\sqrt{\lambda}}{2}\right)}\frac{\pi^2}{6} + o(1/L)\,. \tag{5.15}$$

We now consider the holes contribution in (3.24):

$$-\sum_{h=1}^{H}\int_{-\infty}^{\infty} dk\; 2\pi\, e^{ikx_h}\,\frac{J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\sqrt{\lambda}\;k\cosh\frac{k}{2}}\,. \tag{5.16}$$

From (3.8) we deduce again the leading behaviour

$$x_h \sim \frac{2I_h+\delta-1}{L}\,. \tag{5.17}$$

It follows that the holes contribution to the energy is

$$-H\int_{-\infty}^{\infty} dk\; 2\pi\,\frac{J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\sqrt{\lambda}\;k\cosh\frac{k}{2}} + \mathcal{O}(1/L^2) = -H\pi\sum_{l=0}^{\infty}\frac{(-\lambda)^l}{2^{4l}l!(l+1)!}|E_{2l}| + \mathcal{O}(1/L^2)\,, \tag{5.18}$$

where E_{2l} are the Euler numbers.

Therefore, summing up (5.15), (5.18) with the leading contribution to (3.24) proportional to L, we get that the energy of the multi-loop chain in the limit $L\to\infty$ behaves as follows

$$\begin{aligned} E = {} & \frac{8\pi}{\sqrt{\lambda}}L\int_0^{\infty} dk\;\frac{J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)J_1\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{k(e^k+1)} - H\pi\sum_{l=0}^{\infty}\frac{(-\lambda)^l}{2^{4l}l!(l+1)!}|E_{2l}| + \\ & + \frac{4i}{L\sqrt{\lambda}}\frac{J_1\left(-i\frac{\sqrt{\lambda}}{2}\right)}{J_0\left(i\frac{\sqrt{\lambda}}{2}\right)}\frac{\pi^2}{6} + o(1/L)\,. \end{aligned} \tag{5.19}$$

The SO(6) **symmetric chain.** In an analogous way, one can estimate exactly the coefficient of the $1/L$ correction to the energy (4.16) of the SO(6) chain (as $L\to\infty$). This correction is contained in the two integrals of (4.16)

$$\begin{aligned} \Delta E(L) &= \sum_{i=1}^{2}\int_{-\infty}^{\infty} dy\, E_i(y)\,\mathrm{Im}\ln\left[1+e^{iZ_i(y+i0)}\right] \\ &= \sum_{i=1}^{2}\int_{-\infty}^{0} dy\, E_i(y)\,\mathrm{Im}\ln\left[1+e^{iZ_i(y+i0)}\right] + \\ &\quad + \sum_{i=1}^{2}\int_{0}^{\infty} dy\, E_i(y)\,\mathrm{Im}\ln\left[1+e^{iZ_i(y+i0)}\right]. \end{aligned} \tag{5.20}$$

As in the SU(2) case, we perform different changes of variables according to the region of integration,

$$y = \theta - \frac{2}{\pi} \ln 2L \,, \quad \text{for } y < 0 \,, \quad y = \theta + \frac{2}{\pi} \ln 2L \ , \quad \text{for } y > 0 \ ,$$

and then we let $L \to \infty$. In this limit we get

$$\begin{aligned}\Delta E(L) = & \frac{\pi}{L} \int_{-\infty}^{\infty} d\theta \left\{ \frac{1}{\sqrt{2}} e^{\frac{\pi\theta}{2}} \operatorname{Im} \ln \left[1 + e^{iZ_1^-(\theta+i0)} \right] + e^{\frac{\pi\theta}{2}} \operatorname{Im} \ln \left[1 + e^{iZ_2^-(\theta+i0)} \right] \right\} + \\ & + \frac{\pi}{L} \int_{-\infty}^{\infty} d\theta \left\{ -\frac{1}{\sqrt{2}} e^{-\frac{\pi\theta}{2}} \operatorname{Im} \ln \left[1 + e^{iZ_1^+(\theta+i0)} \right] - e^{-\frac{\pi\theta}{2}} \operatorname{Im} \ln \left[1 + e^{iZ_2^+(\theta+i0)} \right] \right\} + \\ & + o(1/L) \,, \end{aligned} \tag{5.21}$$

where $o(1/L)$ indicates terms that vanish as (5.3). The new functions

$$Z_1^{\mp}(\theta) = \lim_{L\to\infty} \left[Z_1 \left(\theta \mp \frac{2}{\pi} \ln 2L \right) \pm L\pi \right] \,, \quad Z_2^{\mp}(\theta) = \lim_{L\to\infty} \left[Z_2 \left(\theta \mp \frac{2}{\pi} \ln 2L \right) \pm L\frac{\pi}{2} \right] \,,$$

satisfy the *kink* NLIEs:

$$\begin{aligned} Z_1^{\mp}(\theta) = & \pm\sqrt{2} e^{\pm\frac{\pi\theta}{2}} + 2 \int_{-\infty}^{\infty} d\theta' \ G_{11}(\theta-\theta') \operatorname{Im} \ln \left[1 + e^{iZ_1^{\mp}(\theta'+i0)} \right] + \\ & + 2 \int_{-\infty}^{\infty} d\theta' \ G_{12}(\theta-\theta') \operatorname{Im} \ln \left[1 + e^{iZ_2^{\mp}(\theta'+i0)} \right] \,, \\ Z_2^{\mp}(\theta) = & \pm e^{\pm\frac{\pi\theta}{2}} + 2 \int_{-\infty}^{\infty} d\theta' \ G_{21}(\theta-\theta') \operatorname{Im} \ln \left[1 + e^{iZ_1^{\mp}(\theta'+i0)} \right] + \\ & + 2 \int_{-\infty}^{\infty} d\theta' \ G_{22}(\theta-\theta') \operatorname{Im} \ln \left[1 + e^{iZ_2^{\mp}(\theta'+i0)} \right] \,. \end{aligned} \tag{5.22}$$

Now, it happens that the two contributions of order $1/L$ in (5.21) can be exactly computed by generalizing the derivative lemma to the SO(6) case — a case with two coupled NLIEs. These two contributions are equal and their sum gives the order $1/L$ contribution to the energy:

$$\Delta E(L) = \frac{\pi^2}{2L} + o(1/L) \,. \tag{5.23}$$

We conclude that in the limit $L \to \infty$ the energy (4.16) of the anti-ferromagnetic state of the SO(6) chain is given by

$$E = 2L \left(\frac{\pi}{2} + \ln 2 \right) + \frac{\pi^2}{2L} + o(1/L) \,. \tag{5.24}$$

This finite size correction induces to think of a $c = 3$ two-dimensional conformal field theory.

6. Numerical analysis

In the previous section we have computed the explicit contribution to the energy up to order $1/L$ for growing L. However, the NLIE formulation of the Bethe Ansatz equations

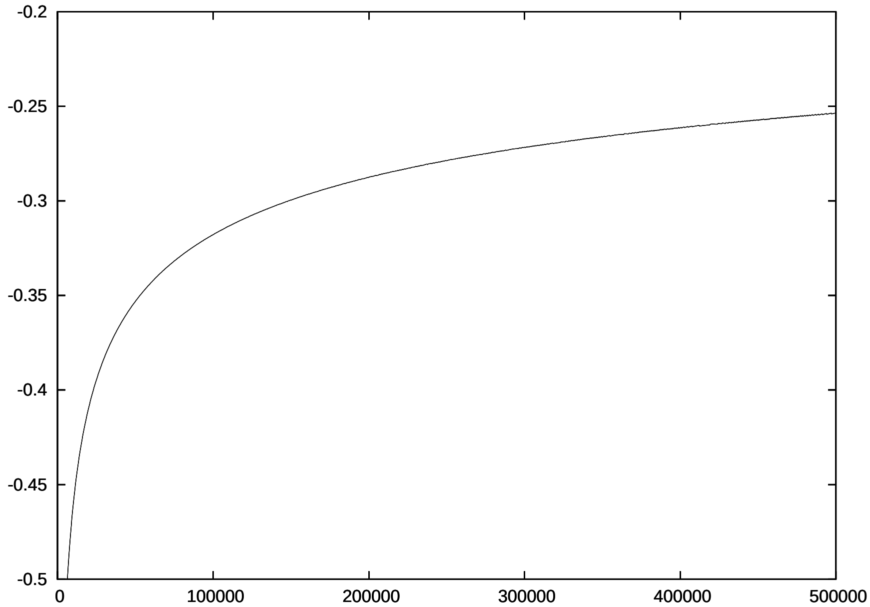

Figure 1: Plot of $\Delta E(L)L - \frac{\pi^2}{6}$ versus L for the state $I = (-1, 0, 1, 2)$ of the Heisenberg spin chain.

allows us to work out interesting and precise numerical computations as well (cf. [33, 38] as first numerical results within this approach). The latter can be used, for instance, to confirm and to improve analytical results. In this spirit, we have performed numerical calculations in order to study contributions to the energy of orders equal to and smaller than $1/L$, when L is very large.

We show here few examples of our numerical solutions of the equations. The most natural method is to solve them by iterations [38]. Even if the obtained equations are correct for all numbers of holes and lengths of the chain, they are particularly effective if one considers states with a small number of holes in a 'Fermi-Dirac sea' of real roots.

Heisenberg chain. We consider a zero momentum state with four holes quantised by $I = (-1, 0, 1, 2)$ (see also figure 2 for a prototypical example of the behaviour of the counting function) and we follow the evolution of E with L. The goal is to show the order of contribution of the various terms in (2.33) when L is large. The leading contribution is explicit: $2L \ln 2$. From the discussion in the previous section, we know that the holes depending term gives a contribution $-H\pi + \mathcal{O}(1/L^2)$. The contribution in $1/L$ is contained in the integral term (5.1) that behaves as

$$\Delta E(L) = \frac{\pi^2}{6L} + o(1/L)\,, \tag{6.1}$$

where $o(1/L)$ indicates corrections that vanish faster than $1/L$. In trying to get some insight on them we extended our analysis to chains of up to five millions sites because we observed that the quantity $\Delta E(L)\,L - \frac{\pi^2}{6}$ is extremely slow to vanish at growing L, as is apparent in figure 1.

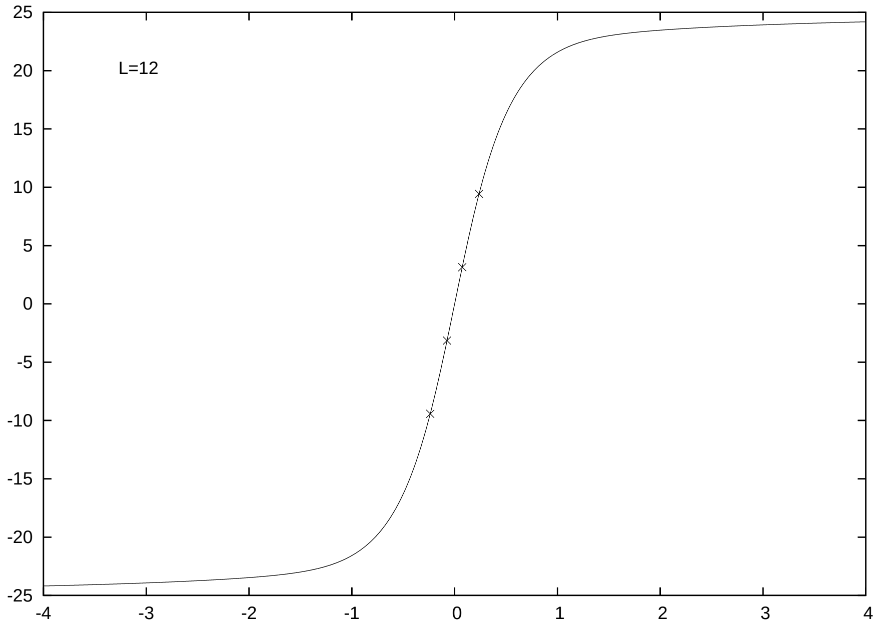

Figure 2: Plot of the counting function $Z(x)$ versus x for the Heisenberg spin chain with $L = 12$ sites. The position of the four holes quantised by $I = (-1, 0, 1, 2)$ is indicated by the small crosses.

This behaviour would be perfectly consistent with the presence of logarithmic terms [44] (inside $o(1/L)$). Further, we expect the first one of them to take the following form

$$\Delta E(L)\, L - \frac{\pi^2}{6} = \frac{c_1}{\ln^2 L} + \cdots \tag{6.2}$$

and we have found a good agreement with the data coming from the numerical solution of the NLIE. However, we refrain from giving any estimate of the constant c_1, because numerical computations for very large chains are technically difficult and are affected by growing numerical errors. We expect to produce more precise data in the future to better understand the additional terms in (6.1).

The finite size corrections to the anti-ferromagnetic vacuum of the Heisenberg chain have been extensively studied in condensed matter literature [44, 36]. Taking into account the first logarithmic corrections we have

$$E = 2L\, \ln 2 + \frac{\pi^2}{6L}\left(1 + \frac{3}{8}\frac{1}{\ln^3 L} + k_2 \frac{\ln\ln L}{\ln^4 L} + \frac{k_3}{\ln^4 L}\right) + \cdots\,, \tag{6.3}$$

where the numerical values of the constants k_2 and k_3 have been obtained by a fit of the numerical data and have been found to be in agreement with those of Karbach and Mütter [45]. We remark that logarithmic corrections to the energy of the anti-ferromagnetic state are smaller than those for states containing holes. This seems to be a general property of this state.

SU(2) **at many-loops.** We can perform similar investigations on $\Delta E(L, \lambda)$ in (5.9) when L is large. Even in this case, in the presence of holes we immediately see that $o(1/L)\, L$

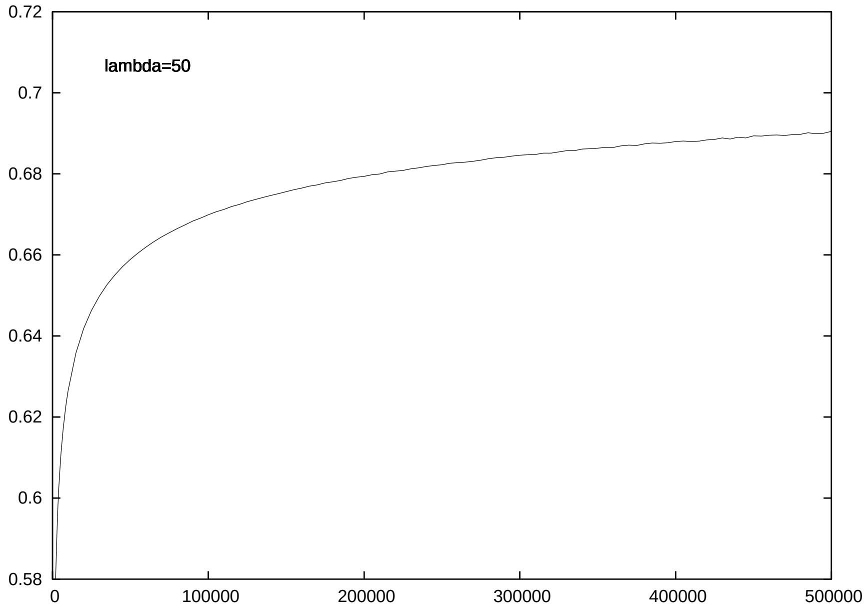

Figure 3: Plot of $\Delta E(L,\lambda)\,L$ versus L for the state $I=(-1\,,0\,,1\,,2)$ of the many-loops spin chain with $\lambda=50$.

vanishes very slowly with L and this suggests the presence of logarithmic corrections. As pointed out in [36] this logarithmic behaviour is related to the decay at infinity of the kernel as a power. Since the multi-loop Bethe equations of [7] provide the same kernel as the Heisenberg chain, the presence of such logarithmic corrections is somehow expected.

We used numerical data, obtained for the same four holes state we used in the Heisenberg case and for $\lambda=50$, to test the agreement with the following guess for the logarithmic corrections

$$\Delta E(L,\lambda)\,L - 0.7843670037\ldots = \frac{b_1}{\ln^2 L} + \cdots \tag{6.4}$$

(on the left hand side we have provided the numerical value of the coefficient of $1/L$ in (5.15) for $\lambda=50$). As in the Heisenberg case, we refrain from giving any estimate of the constant b_1, because of the growing numerical errors that affect computations for very large chains and we postpone them to a forthcoming publication [43].

However, in close analogy with the Heisenberg chain, the proposed functional form (6.4) for the logarithmic corrections was found to be in good agreement with the numerical solution of the NLIE. In figure 3 it is possible to find the plot of the data for $\Delta E(L,\lambda)\,L$ which have been used for our analysis.

Finally, we guessed that the finite size corrections for $E(L,\lambda)$ in the case of the anti-ferromagnetic state will have the same structure of those corresponding to the Heisenberg chain

$$E(L,\lambda) = \frac{8\pi}{\sqrt{\lambda}} L \int_0^\infty dk\, \frac{J_0\!\left(\frac{\sqrt{\lambda}}{2\pi}k\right) J_1\!\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{k(e^k+1)} + \frac{4i}{L\sqrt{\lambda}} \frac{J_1\left(-i\frac{\sqrt{\lambda}}{2}\right)}{J_0\left(i\frac{\sqrt{\lambda}}{2}\right)} \frac{\pi^2}{6} + o(1/L)\,,$$

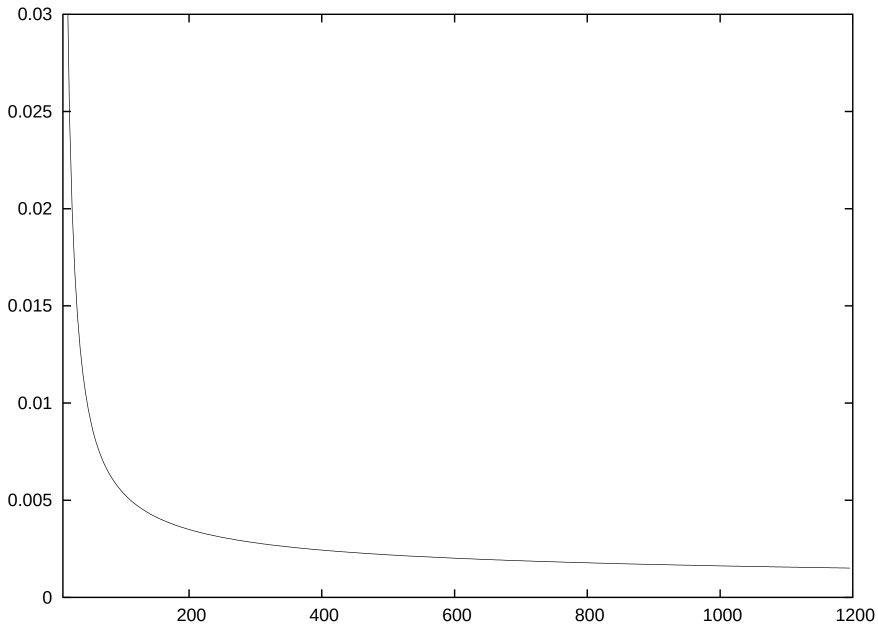

Figure 4: Plot of $\Delta E(L)\,L - \frac{\pi^2}{2}$ versus L for the anti-ferromagnetic state of the SO(6) spin chain.

with

$$o(1/L) = \frac{c(\lambda)}{L \ln^3 L} + \cdots \ . \tag{6.5}$$

Again, in the case $\lambda = 50$, we obtained a good agreement between the previous expression and the numerical data.

The SO(6) **chain.** The same procedure used for the SU(2) case applies to the SO(6) chain, the main difference being that now we need to solve two coupled equations. This does not imply any additional difficulty in their solution, but simply increases the computational time. As before, we investigate $\Delta E(L)\,L$ for large L and compare numerical results with the analytical evaluation (5.23), according to which

$$\Delta E(L)\,L - \frac{\pi^2}{2} = L\,o(1/L)\,. \tag{6.6}$$

Thanks to the various insights from the above-discussed cases, we may for now limit our analysis to chains with up to $L = 1200$ sites: we intend to improve the details in an incoming paper [43] along with analytic supports. Nevertheless, although not completely definitive, the leading behaviour of $L\,o(1/L)$ seems to repeat that of the Heisenberg ground state, namely $1/\ln^3 L$. The corresponding picture is in figure 4.

7. Conclusive remarks

Briefly said, the novelty of this work may be summarised in the derivation of the NLIEs and their application to computing anomalous dimensions and momenta in $\mathcal{N} = 4$ SYM theory when the number of operators, L, is finite. More specifically, our attention has been focussed on the SU(2) scalar sub-sector for many loops and the more general SO(6) scalar sector for one loop. And for what concerns the SU(2) case we started by the conjectured Bethe equations of [7], which are believed to be trustable at least up to order g^{2L-2}. Actually, we have also re-derived, following our own route, the know formulæ for the isotropic spin 1/2 XXX Heisenberg chain as finite L results for the one loop correction in the SU(2) scalar sub-sector: this effort was at least useful to warm up and check the entire machinery.

These NLIEs, equipped with the quantisation conditions for the holes, are totally equivalent to the Bethe equations we started with. But of course they are much more effective for both the analytic and numeric computations. For simplicity's sake, we limited our analysis to the case of real roots, although the introduced formalism can be easily extended to allow for the possible presence of complex roots, as initiated by [33, 38]. Of course, because of its intrinsic nature this NLIE idea is of easier applicability in presence of a very large number of real roots, a limited number of holes and possibly of complex roots. In terms of the spin chain this corresponds to focus the attention on the anti-ferromagnetic state and excitations over it. However, the anti-ferromagnetic configuration may receive particular interest in SYM theory as that with the largest anomalous dimension; so, in this respect it is antipodal to the ferromagnetic vacuum.

Nonetheless, the present formulation gives rise to compact and exact expressions for the 'observables' valid for any L. For instance, the anomalous dimension assumes a form which, in principle, could be exactly computed after solving the NLIE for the counting function $Z(x)$ and after fixing the hole positions, x_h, by the quantization conditions. In this respect, the analytical results of section 5 have given explicitly the anomalous dimension up to the order $1/L$ (in the limit $L \to \infty$). In addition, the numerical work of section 6 has also shown the next-to-leading logarithmic dependence (on L). In principle, the same ingredients might also be exploited to obtain all the other conserved charges underlying integrability [35]. Therefore, a comparison with the string theory integrability, first disclosed by [46], would be highly instructive and desirable.

On the other hand, we would like to remark that the method used here is quite flexible and can be applied to various integrable models. Among them it is important to mention the Hubbard model for its recent relevance in the computation of anomalous dimensions of SYM [16]. In this respect, it is not clear whether the Hubbard chain might be the *ultimate* model in the SYM/IM relation since there seems to be no trace of such a model (or its symmetries) on the string side (not to say on the SYM side).

Finally, it will be interesting to apply the present approach to the case of large-N QCD [11], where the integrable anti-ferromagnetic spin-1 XXX chain appears in the computation of anomalous dimensions.

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